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DETERMINATION OF REFLECTED AND TRANSMITTED FIELDS BY GEOMETRICAL OPTICS

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ABSTRACT

By an extension of ordinary geometrical optics (or accoustics) the intensity of the reflected and transmitted fields due to a point source in the presence of an arbitrary interface between two media is found. A particular consequence of the solution is the general lens and mirror law and the equations for the caustic surfaces.

1. Introduction

The calculation of the electromagnetic or acoustic field at any point in space when two different media are present necessitates a solution of Maxwell's equations or the wave equation of acoustics with appropriate boundary conditions in each case. Exact solutions have been obtained only for the simplest configurations[1,2] and approximate solutions have been found by special means in some other cases (e.g. when one of the media is an arbitrarily curved thin shell[3,4]).

A general procedure for obtaining an approximate solution to an electromagnetic or acoustic problem is the method of geometrical optics or acoustics. By this it is meant that the field quantities propagate along rays which are determined by the Fermat principle of least time, and that these rays obey the laws of reflection and refraction at the interface between two different media. Furthermore, in this method it is also possible to determine the field components themselves[5,6] - something which has not ordinarily been done in geometrical optics or acoustics. Thus, for example, the reflected and transmitted field components at an interface are related to the incident field components by the well known Fresnel formulae or the corresponding formulae in acoustics. Also the field components vary along a ray inversely as the square root of the velocity and of the area of the normal cross section of an infinitesimal tube of rays containing the ray in question. These results enable one to determine the geometrical optics or acoustics field completely.** It is to be expected that this solution will be an adequate approximation to the full solution only at very high frequencies (i.e. at wavelengths small compared to the dimensions of the problem).*

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* As a matter of fact R. K. Luneberg has shown quite generally that the first terms in asymptotic expansions of the space dependence of the electric and magnetic field vectors is that given by geometrical optics. This material will appear in a forthcoming report.

**Phase is determined by the path length along a ray.

In the present investigation an incident field due to a point source is assumed to impinge upon an arbitrarily curved or flat interface which separates two homogeneous and isotropic media. The magnitudes of the reflected and transmitted field components at any point are determined on the basis of geometrical optics, i.e. by applying the theory outline above. For the case of reflection the problem has already been solved[3,7], but it is included here because this can be done with no additional difficulty. The transmission problem has been treated by the Kirchhoff method[3,8] and the present calculation constitutes a check on that solution. The check for the Kirchhoff method applied to reflection is contained here as well as in reference 3.

2. Formulation of the Problem

A point source is assumed to be located at a point (x_1, y_1, z_1) in a homogeneous isotropic medium with propagation speed V_I. The surface $z = z(x,y)$ separates this medium from a different homogeneous isotropic medium with propagation speed V_T.

The magnitudes of the reflected and transmitted field components at the surface can be found in terms of the incident field components and the Fresnel formulae (only for those surface points which can be connected to the source by a straight line segment lying in the first medium). Let $E(x,y,z)$ represent the amplitude of any field component at the point (x,y,z) due to reflection or transmission from the surface. Then from geometrical optics[5] we have the relation

$$E(x,y,z) = E_i(x',y',z')\sqrt{\frac{d\sigma'}{d\sigma}}$$

where (x',y',z') is the point in which a reflected or transmitted ray through (x,y,z) intersects the surface $z' = z(x',y')$. $E_i(x',y',z')$ is the reflected or transmitted field component at the surface, where $i = 1$ for reflection and $i = 2$ for transmission. These components can be calculated, as previously stated, from the incident field component at the

surface. The quantity $d\sigma$ is the area in which an infinitesimal tube of reflected or transmitted rays containing the ray through (x,y,z) and (x',y',z') cuts a plane p normal to this ray at the point (x,y,z), (see Fig. 2). $d\sigma'$ is the area enclosed by this same tube of rays on a plane p' normal to the ray in question at the point (x',y',z'). In the limit of infinitesimal areas the ratio $d\sigma'/d\sigma$ is just the Jacobian, $J(\frac{p'}{p})$, of the transformation established by reflected or transmitted rays, which maps the plane p' into the plane p. Hence we may write

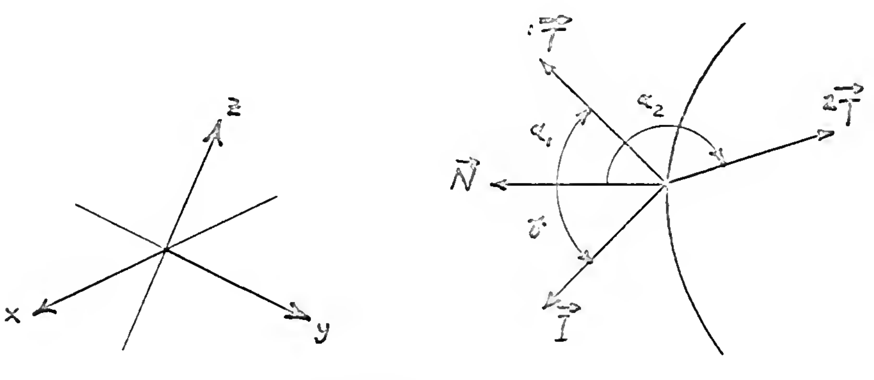
$$(1) \qquad E(x,y,z) = J^{1/2}\left(\frac{p'}{p}\right) E_1(x',y',z') \quad .$$

3. Calculation of the Jacobian

Let $\vec{N}(x',y',z')$ be a unit vector normal to $z' = z(x',y')$ and pointing into the first medium; $\vec{I}(x',y',z')$ is the unit vector pointing from (x',y',z') to the source at (x_1,y_1,z_1). The direction of the reflected ray at (x',y',z') is given by the unit vector ${}^{1}\vec{T}(x',y',z')$ and that of the transmitted ray by ${}^{2}\vec{T}(x',y',z')$ where, from the laws of reflection and refraction [see Appendix] :

$$(2) \quad {}^{i}\vec{T}(x',y',z') = -n^{i-1}\vec{I} + \left[n^{i-1}(\vec{I}\cdot\vec{N}) - (-1)^{i}\left\{1-n^{2(i-1)}\left[1-(\vec{I}\cdot\vec{N})^2\right]\right\}^{1/2}\right]\vec{N}. \qquad (i = 1,2)$$

Here $n = \dfrac{\sin \alpha_i}{\sin \tau}$ and $i = 1$ for reflection, $i = 2$ for refraction or transmission. The angles and unit vectors are shown in Fig. 1.



(x_1, y_1, z_1)

Fig. 1

The unit vectors $\vec{I}$ and $\vec{N}$ have the components:

$$\vec{I}(x',y',z') = \left(\frac{x_1 - x'}{\ell(x',y',z')} , \frac{y_1 - y'}{\ell(x',y',z')} , \frac{z_1 - z'}{\ell(x',y',z')} \right)$$

(3)

$$\vec{N}(x',y',z') = \left(\frac{-\frac{\partial z'}{\partial x'}}{\mathcal{n}(x',y',z')} , \frac{-\frac{\partial z'}{\partial y'}}{\mathcal{n}(x',y'z')} , \frac{1}{\mathcal{n}(x',y',z')} \right),$$

where:

$$\ell^2(x',y',z') = (x_1 - x')^2 + (y_1 - y') + (z_1 - z')^2$$

and

$$\mathcal{n}^2(x',y',z') = 1 + (\frac{\partial z'}{\partial x'})^2 + (\frac{\partial z'}{\partial y'})^2$$

The coordinate system must be so chosen that $\vec{N}$ points into the first medium.

For an arbitrary point (x',y',z') on the surface the equation of the reflected or transmitted ray through this point is

$$\frac{x - x'}{{}^i T_x} = \frac{y - y'}{{}^i T_y} = \frac{z - z'}{{}^i T_z}$$

and thus:

$$x = x' + (z - z') \frac{{}^i T_x(x',y',z')}{{}^i T_z(x',y',z')}$$

(4)

$$y = y' + (z - z') \frac{{}^i T_y(x',y',z')}{{}^i T_z(x',y',z')}$$

where the subscripts indicate components of the vector ${}^i\vec{T}$.

For a fixed value of z equations (4) are the mapping of the surface S given by $z' = z(x',y')$ onto the plane π, given by z = const (see Fig. 2). From these equations we can compute the Jacobian of this transformation, namely,

(5) $$J(\frac{S}{\pi}) = J^{-1}(\frac{\pi}{S}) = \left\{ \frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'} \frac{\partial y}{\partial x'} \right\}^{-1}$$

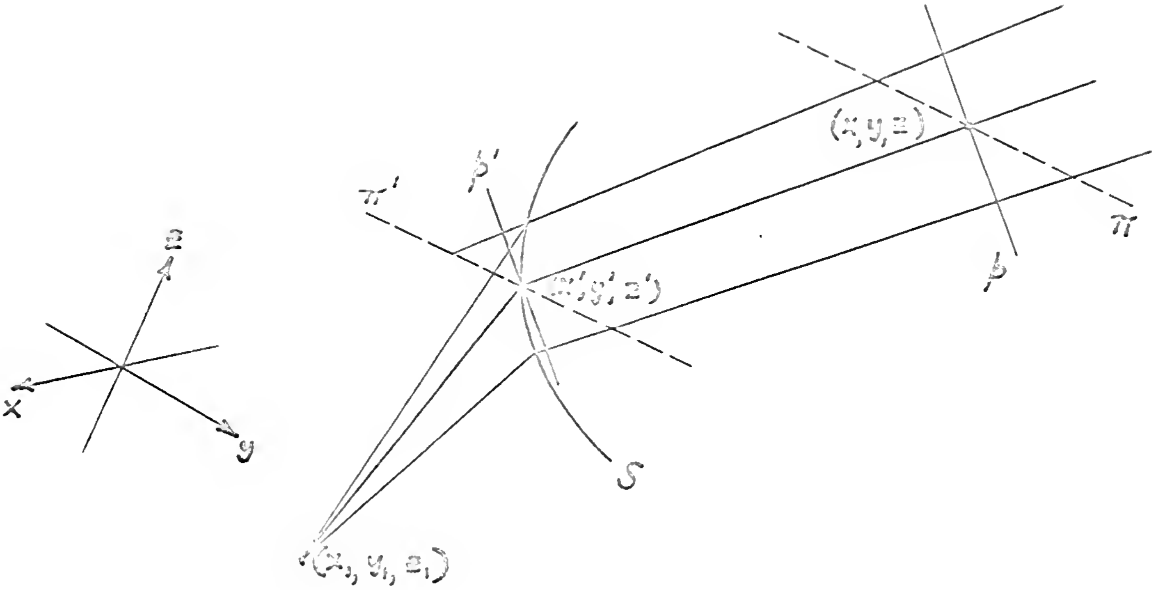


Fig. 2 (Transformations for Transmission)

The Jacobian which occurs in equation (1) can now be calculated since

$$J\left(\frac{p'}{p}\right) = J\left(\frac{p'}{\pi'}\right) J\left(\frac{\pi'}{s}\right) J\left(\frac{s}{\pi}\right) J\left(\frac{\pi}{p}\right) \quad ,$$

where π' is a plane through (x',y',z') and parallel to π, p and p' are the normal planes previously defined and the notation represents the Jacobians of the transformations between indicated surfaces determined by means of the reflected or transmitted rays. However, $J\left(\frac{p'}{\pi'}\right) = J^{-1}\left(\frac{\pi}{p}\right)$ since the planes are parallel by pairs and thus corresponding angles used in defining the transformations are equal. Thus we have:

$$J\left(\frac{p'}{p}\right) = J\left(\frac{\pi'}{s}\right) J\left(\frac{s}{\pi}\right)$$

Now place the origin of coordinates at the point (x',y',z') with the z-axis normal to S and positive into the first medium. Further let the x- and y-axes be parallel to the directions of principal curvature[9] at this point. The plane π' now becomes the xy plane (z=0) and is the tangent plane to S at the new origin. The equation of the surface S expanded into a Taylor series around the origin becomes[9]

(6) $$z' = a(x')^2 + b(y')^2 + \ldots\ldots. \quad ,$$

where 2a and 2b are the principal curvatures of the surface at the origin and the primes denote the coordinates of a surface point relative to the new coordinate system.

In these coordinates $J(\frac{\pi'}{s})$ is the Jacobian of the mapping, by reflected or transmitted rays, of the surface on the xy plane. This mapping is given by

$$(7) \qquad \begin{aligned} x &= x' - z' \tan \theta(x',y') \\ y &= y' - z' \tan \phi(x',y') \end{aligned}$$

where $\theta(x',y')$ and $\phi(x',y')$ are the angles between a ray reflected (or transmitted) at (x',y',z') and the x- and y-axes, respectively. From equations (6) and (7) we can compute

$$J(\frac{\pi'}{s}) = \frac{\partial x}{\partial x'}\frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'}\frac{\partial y}{\partial x'} \quad .$$

We find that, at the origin (i.e. $x' = y' = z' = 0$), the above Jacobian is equal to one. Using this result in the preceding expression for $J(\frac{p'}{p})$ we obtain

$$J(\frac{p'}{p}) = J(\frac{s}{\pi}) \ .$$

With the x-,y-,z-axes chosen as above we may calculate, by using equations (4) in equation (5), the above Jacobian:

$$J(\frac{p'}{p}) = J(\frac{s}{\pi}) = \left\{1 + z\left[\frac{\partial}{\partial x'}\left(\frac{{}^iT_x}{{}^iT_x}\right) + \frac{\partial}{\partial y'}\left(\frac{{}^iT_y}{{}^iT_z}\right)\right]\right.$$

$$\left. + z^2\left[\frac{\partial}{\partial x'}\left(\frac{{}^iT_x}{{}^iT_z}\right)\frac{\partial}{\partial y'}\left(\frac{{}^iT_y}{{}^iT_z}\right) - \frac{\partial}{\partial y'}\left(\frac{{}^iT_x}{{}^iT_z}\right)\frac{\partial}{\partial x'}\left(\frac{{}^iT_y}{{}^iT_z}\right)\right]\right\}^{-1} \quad ;$$

where the partial derivatives are to be evaluated at the origin. The components of the vector ${}^i\vec{T}$ and their partial derivatives evaluated at the origin are:

$${}^{i}T_z = -(-1)^i \left\{1 - n^{2(i-1)}\left[1 - \frac{z_1^2}{D_1^2}\right]\right\}^{1/2} = \cos\alpha_i$$

$${}^{i}T_x = -\,n^{i-1}\, x_1/D_1$$

$${}^{i}T_y = -\,n^{i-1}\, y_1/D_1$$

$$\frac{\partial\,{}^{i}T_x}{\partial x} = \frac{n^{i-1}}{D_1}\left(1 - \frac{x_1^2}{D_1^2}\right) - 2a\,A \;, \quad \frac{\partial\,{}^{i}T_x}{\partial y} = -n^{i-1}\,\frac{x_1 y_1}{D_1^3}$$

$$\frac{\partial\,{}^{i}T_y}{\partial y} = \frac{n^{i-1}}{D_1}\left(1 - \frac{y_1^2}{D_1^2}\right) - 2b\,A \;, \quad \frac{\partial\,{}^{i}T_y}{\partial x} = -\,n^{i-1}\,\frac{x_1 y_1}{D_1^3}$$

(9)
$$\frac{\partial\,{}^{i}T_z}{\partial x} = \left[-\,2a\,An^{i-1}\,\frac{x_1}{D_1} + n^{2(i-1)}\,\frac{x_1 z_1^2}{D_1^4}\right]\frac{1}{\cos\alpha_i}$$

$$\frac{{}^{i}T_z}{y} = \left[-2b\,An^{i-1}\,\frac{y_1}{D_1} + n^{2(i-1)}\,\frac{y_1 z_1^2}{D_1^4}\right]\frac{1}{\cos\alpha_i}$$

where we have used:

$$A = n^{i-1}\,\frac{z_1}{D_1} - (-1)^i\left\{1 - n^{2(i-1)}\left[1 - \frac{z_1^2}{D_1^2}\right]\right\}^{1/2}$$

(10)
$$= n^{i-1}\cos\gamma + \cos\alpha_i \;;$$

$$D_1^2 = x_1^2 + y_1^2 + z_1^2$$

$$= \frac{z_1^2}{\cos^2\gamma}$$

and the definition of n. With the aid of equations (9) the Jacobian in

(8) becomes

$$J\left(\frac{p'}{p}\right)=\left\{1+\frac{z}{\cos\alpha_i}\left[\frac{n^{i-1}}{D_1}\left(1+\frac{z_1^2}{D_1^2}+\frac{n^{2(i-1)}z_1^2(x_1^2+y_1^2)}{D_1^4\cos^2\alpha_i}\right)-2(a+b)A\right.\right.$$

$$\left.-2\left(\frac{ax_1^2+by_1^2}{D_1^2}\right)\frac{n^{2(i-1)}}{\cos^2\alpha_i}\,A\right]$$

$$+\frac{z^2}{\cos^2\alpha_i}\left[\frac{n^{2(i-1)}}{D_1^2}\left(\frac{z_1^2}{D_1^2}+\frac{n^{2(i-1)}z_1^2(x_1^2+y_1^2)}{D_1^4\cos^2\alpha_i}\right)-2(a+b)\left(\frac{n^{i-1}}{D_1}A\right)\right.$$

$$-2\left(\frac{ax_1^2+by_1^2}{D_1^2}\right)\frac{n^{2(i-1)}}{\cos^2\alpha_i}\left(\frac{n^{i-1}}{D_1}A\right)$$

$$\left.\left.+4ab\left(1+\frac{n^{2(i-1)}(x_1^2+y_1^2)}{D_1^2\cos^2\alpha_i}\right)A^2+2\left(\frac{bx_1^2+ay_1^2}{D_1^2}\right)\left(1-\frac{n^{2(i-1)}z_1^2}{D_1^2\cos^2\alpha_i}\right)\left(\frac{n^{i-1}}{D_1}A\right)\right]\right\}^{-1}$$

This equation may be expressed in terms of geometric properties of the surface and distances which are independent of the choice of coordinates. These quantities are[9]

(12)
$$G_m=(a+b)\quad;\quad G_g=4ab$$

$$G_{11}=2\left(\frac{ax_1^2+by_1^2}{D_1^2}\right)\frac{1}{\sin^2\gamma}$$

$$2G_m-G_{11}=2\left(\frac{bx_1^2+ay_1^2}{D_1^2}\right)\frac{1}{\sin^2\gamma}$$

and the distances involved are D_1, already given, and

(13)
$$D^2=x^2+y^2+z^2=\frac{z^2}{\cos^2\alpha_i}\,.$$

G_m and G_g are respectively the mean curvature and the Gaussian curvature of the surface at the point involved; G_{11} is the curvature of the surface in a plane containing the incident ray and the normal to the surface at this point; D_1 is the distance along the incident ray from the source to the surface, and D is the distance along a reflected or transmitted ray from the surface to any point (x,y,z) on this ray. In terms of these invariants equation (11) becomes

$$J\left(\frac{p'}{p}\right) = \left\{ 1 + D\left[\frac{n^{i-1}}{D_1}\left(1 + \frac{\cos^2\gamma}{\cos^2\alpha_i}\right) - (2G_m + G_{11}\tan^2\alpha_i)(n^{i-1}\cos\gamma + \cos\alpha_i)\right]\right.$$

(14)

$$+ D^2\left[\frac{n^{2(i-1)}}{D_1^2}\left(\frac{\cos^2\gamma}{\cos^2\alpha_i}\right) - (2G_m + G_{11}\tan^2\gamma)\left(\frac{\cos^2\gamma}{\cos^2\alpha_i}\right)\left(n^{i-1}\cos\gamma + \cos\alpha_i\right)\frac{n^{i-1}}{D_1}\right.$$

$$\left.\left. + G_g\sec^2\alpha_i(n^{i-1}\cos\gamma + \cos\alpha_i)^2\right]\right\}^{-1} .$$

Now with the origin at any point in space, let the source be at (x_1, y_1, z_1), let a point on the surface be (x', y', z'), and finally let the observation point be (x,y,z) in the new coordinates. Equation (14) is unchanged in these coordinates provided that

$$D_1^2 = (x_1 - x')^2 + (y_1 - y') + (z_1 - z')^2$$

(15)

$$D^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$= \frac{(z-z')^2}{\cos^2\alpha_i} .$$

If equation (14) is used in equation (1) the reflected or transmitted field amplitude is given by:

(16) $E(x,y,z) = E_i(x',y',z')\, J^{1/2}\left(G_m(x',y'), G_g(x',y'), G_{11}(x',y'), \alpha_i, \gamma, D\right)$

since we can express x and y in terms of x', y' and z by means of equations (4). In equation (16) J is a functional symbol for the expression on the right side of equation (14) with the changes noted in (15).

4. Discussion of the Solution

It is of interest to investigate the "level surfaces," that is surfaces on which a reflected or transmitted field component has a constant value. To find such surfaces we require that

$$E(x,y,z) = E_* = \text{const.}$$

in equation (16). We then solve the resulting equation for D and obtain:

$$(17)\quad D(x,y,z) = \left\{1 - \left(\frac{E_i^2(x',y',z')}{E_*^2}\right)\right\}\left\{-\left[\frac{H}{2}\right] \pm \left[\left(\frac{H^2}{4} - K\right) + \frac{E_i^2(x',y',z')}{E_*^2}K\right]^{1/2}\right\}^{-1}$$

where H and K are the coefficients of D and D^2 respectively in equation (14). Equations (4) and (17) are equations for the level surfaces in terms of the parameters x' and y'. For each value of E_* there are two such surfaces as the double sign in equation (17) indicates.

Those level surfaces on which E is infinite are called caustics and are of special importance. Their equations are easily obtained by letting $E_* \to \infty$ in the equations for the level surfaces. The distance from a caustic to the surface along a reflected or transmitted ray is useful. This distance is given by (17) with $E_* = \infty$ and is:

$$(18)\qquad D = \left\{-\frac{H}{2} \pm \sqrt{\left(\frac{H}{2}\right)^2 - K}\right\}^{-1} ;$$

thus each reflected or transmitted ray intersects two caustics.

The two points along a reflected or transmitted ray at which $E(x,y,z) = \infty$ are called conjugate points. If the caustic surfaces intersect then the conjugate points on the rays through the intersection

coincide, and a point image is formed; the image is called real if $D > 0$ and virtual if $D < 0$. This occurs if the second term in (18) is zero, that is

$$\left[H^2 - 4K\right] \equiv \frac{n^{2(i-1)}}{D_1^2}\left[\frac{\cos^2\gamma}{\cos^2\alpha_i} - 1\right]^2$$

(19)

$$+2\,\frac{n^{i-1}}{D_1}\left(2G_m - G_{11}\tan^2\alpha_i\right)\left[\frac{\cos^2\gamma}{\cos^2\alpha_i} - 1\right](n^{i-1}\cos\gamma + \cos\alpha_i)$$

$$+\left[(2G_m + G_{11}\tan^2\alpha_i)^2 - 4G_g\sec^2\alpha_i\right](n^{i-1}\cos\gamma + \cos\alpha_i)^2 = 0,$$

and corresponds to the following possibilities:

For reflection, $\alpha_i = \gamma$ (these results are also given in ref. 3):

1) $\gamma = \pi/2$; then from eq.(18) $D = - D_1$ and the virtual image is the source.

2) $a = b = 0$; all possible angles for γ and $D = - D_1$ This is the case of reflection from a plane surface.

3) $a \geq b > 0$, $a \leq b < 0$, $\cos\gamma = \sqrt{b/a}$, $\theta = \pi/2$

4) $b \geq a > 0$, $b \leq a < 0$, $\cos\gamma = \sqrt{a/b}$, $\theta = 0$

For transmission:

1) $n = 1$; then from eq. (18) $D = - D_1$ and the virtual image is the source.

2) $\gamma = 0$, $\alpha_2 = \pi$; then $G_n^2 = G_g$ and the point of transmission is an umbilical point. From equation (18) the distance is given by

$$\frac{1}{D} = -\frac{n}{D_1} + 2a(n-1)$$

In all of the above cases the distance to the image is expressed by the general formula

(20) $$\frac{1}{D} = -\frac{n}{D_1} \pm \left[1 - (-1)^i n\right] G_g^{1/2}$$

where the sign to be chosen is that of G_n. If the source is at an infinite distance from the reflecting or transmitting surface, the distance of a resulting point image (which is called a focal point) from the surface is called the focal length, f, and is given by equation (20) with $D_1 = \infty$.

(21) $$\frac{1}{f} = \pm \left[1 - (-1)^i n\right] G_g^{1/2} \quad .$$

From equations (20) and (21) we obtain the combined lens and mirror law:

(22) $$\frac{1}{D} + \frac{n}{D_1} = \frac{1}{f} \quad .$$

As another application, we specialize the reflecting surface to a sphere. Then, equation (14) yields the geometrical factor obtained by van der Pol and Bremmer[10] for the reflection of an electromagnetic wave from the earth. When the Fresnel formulae and the phase are taken into account, their complete result (for wavelengths small compared to the earth's radius) is obtained.

APPENDIX

Derivation of Equation for ${}^{i}\vec{T}$

To derive equation (2) of the text proper we use the unit vectors and angles shown in Fig. 1. From the definitions of these vectors and angles we require:

$(\bar{1})$
$$\vec{I}^2 = \vec{N}^2 = {}^{i}\vec{T}^2 = 1$$
$$\vec{I} \cdot \vec{N} = \cos \gamma$$
$$ {}^{i}\vec{T} \cdot N = \cos \alpha_i$$

By the law of reflection $\gamma = \alpha_1$, or:

$(\bar{2})$
$$ {}^{1}\vec{T} \cdot \vec{N} = \vec{I} \cdot N \; ;$$

and from Snell's law $n = \sin \alpha_2 / \sin \gamma$, or:

$(\bar{3})$
$$ {}^{2}\vec{T} \cdot N = - \left\{ 1 - n^2 \left[1 - (\vec{I} \cdot \vec{N})^2 \right] \right\}^{1/2}$$

where we always take the positive square root in Eq. $(\bar{3})$. Since the reflected or transmitted ray must lie in the plane of the normal to the surface and the incident ray (by the law of reflection and Snell's law) we have

$(\bar{4})$
$$ {}^{i}\vec{T} = A_i \vec{I} + B_i \vec{N} \; .$$

Using equation $(\bar{4})$ in equations $(\bar{1})$ and $(\bar{2})$ we find the two solutions

$$\left. \begin{aligned} A_1 &= -1 \\ B_1 &= 2(\vec{I} \cdot \vec{N}) \end{aligned} \right\} \quad \text{and} \quad \left. \begin{aligned} A_1 &= +1 \\ B_1 &= 0 \end{aligned} \right\} \; .$$

The second set of coefficients reproduces the incident ray and so must be discarded. Then the most general ${}^{1}\vec{T}$ vector is given by:

(5)
$$ {}^{1}\vec{T} = - \vec{I} + 2(\vec{I} \cdot \vec{N}) \, \vec{N}.$$

If equation $(\bar{4})$ is used in equations $(\bar{1})$ and $(\bar{3})$ we get the solutions

$A_2 = -n$ and $A_2 = n$

$B_2 = n(\vec{I} \cdot \vec{N}) - \left\{1 - n^2\left[1 - (\vec{I} \cdot \vec{N})^2\right]\right\}^{1/2}$ $B_2 = -n(\vec{I} \cdot \vec{N}) - \left\{1 - n^2\left[1 - (\vec{I} \cdot \vec{N})\right]\right\}^{1/2}$

The second set of coefficients yield a transmitted ray which lies on the same side of the normal as the incident ray. However, this is in contradiction to the law of refraction and so this solution must be abandoned. Using the first set of coefficients gives the general ${}^{2}\vec{T}$ vector as:

$$(6) \qquad {}^{2}\vec{T} = -n\vec{I} + \left(n(\vec{I} \cdot \vec{N}) - \left\{1 - n^2\left[1 - (\vec{I} \cdot \vec{N})^2\right]\right\}^{1/2}\right)\vec{N}$$

Equations $(\bar{5})$ and $(\bar{6})$ can now be written in the combined form:

$$(7) \qquad {}^{i}\vec{T} = -(n^{i-1})\,\vec{I} + \left(n^{i-1}(\vec{I} \cdot \vec{N}) - (-1)^{i}\left\{1 - n^{2(i-1)}\left[1 - (\vec{I} \cdot \vec{N})^2\right]\right\}^{1/2}\right)\vec{N}$$

which is the desired equation. This can also be expressed by the simple expression:

$$(8) \qquad {}^{i}\vec{T} = (-n^{i-1})\,\vec{I} + (n^{i-1}\cos\gamma + \cos\alpha_{i})\,\vec{N}$$

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